Aversion to health inequalities and priorities in health care

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Abstract

Traditionnally aversion to health inequality is modelled through a concave utility function of health outcomes. However recently Bleichrodt et *al.* (2004) have suggested a "dual" approach based on a distortion of the proportion of patients benefiting from a given health outcome. The purpose of this paper is to analyse how priorities in health care are determined in the framework of each model. It appears that policy implications are very sensitive to the choice of the model that will represent aversion to health inequality.

Introduction

As is well known there exist many similarities between the analysis of decision under risk and the measurement of (income or health) inequalities.

A very good example of this proximity is found in the papers by Rothschild and Stiglitz (1970 and 1971) on the one hand and by Atkinson (1971) on the other hand which were published almost simultaneously in the same journal to deal respectively with the notions of greater risk and that of greater inequality.

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While at the begining the two fields developed along the same lines, an important difference emerged through time however. In the field of decision under risk, the relevance of the expected utility model was pretty quickly challenged and many alternative models of choice were proposed and developed. To the best of our knowledge, no such a large disruption appeared in the field of inequality measurement where the utility approach remained dominant with some exceptions however (see e.g. Weymark (1981), Ebert (1988), Yaari (1988) and Bleichrodt et *al.* (2004)). In this very recent paper, these authors - from now on B. et *al.* - suggest to measure inequality in a way similar to that used in the dual theory of risk (Yaari 1987): instead of transforming the outcomes (e.g. wealth, health or income) through a utility function, B. et *al.* as well Yaari suggest to transform either the probability of the outcome (in risk theory) or the proportion of people in each category of outcome (in B. et *al.*), which is of course quite similar to the transformation of a probability.

Quite frequently, the alternative models (e.g. such as EU versus Yaari's dual theory) are evaluated in terms of the quality of their underlying axioms. However, they may also be evaluated from their implications for choices and decisions[1]. It is precisely the purpose of this paper to compare the usual approach to the measurement of health inequality (the "utility" one) with the "dual approach" suggested by B. et *al.* not in terms of their axioms but in terms of their implications for priority settings in health.

In the next section (section 1) we define the environment of the model which is very close to the one adopted by Hoel (2003) and subsequently in Bui et *al.* (2005) to analyse the impact of treatment risks on health care priorities[2]. Sections 2 and 3 are then devoted respectively to the "utility" and the "dual" approaches of inequality measurement. In the conclusion, we contrast the results obtained in each approach and some potential extensions are suggested.

1 Description of the environment

The population of a country is made of two types of individuals. A proportion α of this population suffers from a severe disease (disease 1) which - if it remains untreated - produces a health level h_1. The rest of the population

[1] A comparison along these lines between the EU model and the dual theory of choice under risk was made by Doherty and Eeckhoudt (1995) for the case of insurance decision.

[2] In appendix 1, we consider a model different from that of Hoel (2003) in order to briefly discuss the sensitivity of the results to the environment.

$(1-\alpha)$ is affected by a minor disease yielding a health stock h_2 $(> h_1)$ if untreated.

For each disease there exists a treatment in which the social planner can invest. This investment per patient is denoted c_1 for disease 1 and c_2 for disease 2. Expanding c_1 or c_2 improves the corresponding health levels in a way characterized by the continuous and everywhere differentiable functions $h_1(c_1)$ and $h_2(c_2)$ with $h_i'(c_i) \geqslant 0$ and $h_i''(c_i) < 0$ for i=1 or 2.

Besides we assume that even if disease 1 were treated to the fullest extent (c_{1m}), i.e. up to the point where $h_1'(c_{1m}) = 0$, the resulting health level $h_1(c_{1m})$ would still be below h_2. In other words, whatever the treatment choices made by a social planner disease 1 remains the worst one[3]. To improve the health of the whole population the social planner avails upon a fixed budget B per person that she has to allocate between c_1 and c_2 . Hence she faces the budget constraint:

$$\alpha c_1 + (1-\alpha)c_2 \ \leq B \tag{1.1}$$

If the social planner is inequality neutral she will try to maximize the total health level defined by:

$$R = \alpha h_1(c_1) + (1-\alpha)h_2(c_2) \tag{1.2}$$

subject to (1.1).

Without surprise the resulting allocation rule is:

$$h_1'(c_1^*) = h_2'(c_2^*) \tag{1.3}$$

when corner solutions do not prevail.

Quite interestingly, health priorities - i.e. the relative importance of c_1 and c_2 at the optimum - for an inequality neutral social planner are not affected by the proportion of persons benefiting from the treatment: they only depend upon the marginal effectiveness of each technology. Besides since only marginal effectiveness matters, an increase in the severity of disease 1 - i.e. if $h_1(c_1)$ becomes $h_1(c_1) - r$ with $r > 0$ - does not change the optimal budget allocation. Obviously these two properties of the optimal allocation by an inequality neutral social planner are not in line with many current policy recommendations and justify the interest for a model including both efficiency and inequality aversion parameters. In the next two sections we consider two such models of inequality aversion.

[3] See the conclusion for comments about this assumption.

2 The utility model

We first consider the case where the social planner expresses her inequality aversion by transforming the health outcomes through an increasing and concave utility function $U(h)$. Her objective then becomes:

$$\mathrm{Max}_{c_1,c_2} \quad R = \alpha U(h_1(c_1)) + (1-\alpha)U(h_2(c_2)) \tag{2.1}$$

under the same budget constraint (1.1)[4] and the associated first order conditions are:

$$\alpha U'(h_1(c_1))h_1'(c_1) - \alpha\lambda = 0$$

$$(1-\alpha)U'(h_2(c_2))h_2'(c_2) - (1-\alpha)\lambda = 0 \tag{2.2}$$

or

$$U'(h_1(\widehat{c}_1))h_1'(\widehat{c}_1) = U'(h_2(\widehat{c}_2))h_2'(\widehat{c}_2) \tag{2.2'}$$

When interior solutions prevail[5], the concavity of U and of the h functions combined with the fact that $h_1(c_1) < h_2(c_2)$ for every pair (c_1, c_2) implies that:

$$\widehat{c}_1 > c_1^*$$

$$\widehat{c}_2 < c_2^* \tag{2.3}$$

Because of her inequality aversion the social planner reallocates funds to the benefit of the more severe disease. Besides, it is easily shown that when the social planner becomes more inequality averse this tendency is reinforced. In the tradition of Pratt (1964), more inequality aversion is captured by a concave transformation of U into V defined by:

$$V = k\left[U\right]$$

with $k' > 0$ and $k'' < 0$.

Using standard arguments we can then easily prove that $\widehat{c}_{1,V} > \widehat{c}_{1,U}$ where the second subscript indicates which utility function is considered.

[4] Notice the similarity between this problem and that of Hoel. The only difference is that here there is no uncertainty about the effect of the treatments.

[5] It is easily checked that the second order condition for a maximum holds.

Of course these results are in line with common sense: the more inequality averse social planner is willing to accept a larger efficiency loss measured by the intensity of the difference between the marginal productivities of c_1 and c_2 .

The utility approach to the measurement of inequality aversion has another desirable feature. Indeed it implies that a deterioration of disease 1 (through the substraction of a constant r from $h_1(c_1)$) induces a reallocation of funds towards disease 1. To prove this, write the social planner's objective as:

$$\text{Max}_{c_1,c_2} \quad R = \alpha U(h_1(c_1) - r) + (1-\alpha)U(h_2(c_2))$$

so that one easily obtains, in accordance with intuition:

$$\frac{d\widehat{c}_1}{dr} > 0 \tag{2.4}$$

As condition 1 deteriorates relative to condition 2, the social planner reallocates funds towards the deteriorated condition in order to partially reduce the difference between the (final) health levels.

While the results presented so far illustrate the relevance of the utility approach to describe policy choices[6], we now turn to its fundamental weakness.

The first order conditions detailed in (2.2) immediately reveal that $\widehat{c}_1$ and $\widehat{c}_2$ are independent of the value of α: the optimal allocation of the budget between the potential diseases is not affected by the number of persons involved. Such a property does not seem to fit two widespread although contradictory opinions: quite a few people claim that priorities should be given to the treatment of diseases affecting a wide population while others think that not enough attention is paid to orphan diseases which concern only a very limited number of patients. Each of these opinions suggests that there should be a link (positive or negative) between c_1 and α.

3 The dual approach to inequality aversion

As suggested by B et *al.,* aversion to health inequality can alternatively be expressed by distorting the proportions of patients involved instead of the health outcomes. As a result the objective function of the social planner is now written as:

[6] Notice however that if we consider a multiplicative deterioration of condition 1 - instead of an additive one - matters are less clear-cut. See appendix 1.

$$\text{Max}_{c_1,c_2}\ R = W(\alpha)h_1(c_1) + (1 - W(\alpha))h_2(c_2) \tag{3.1}$$

where W is an increasing and concave function of α with besides the following properties:

$$W(0) = 0 \text{ and } W(1) = 1 \tag{3.2}$$

It follows from (3.2) and the concavity of $W(\alpha)$ that for any α, $W(\alpha) > \alpha$. Hence R is always below the expected health outcome and the difference between this expected health outcome and R is analogous to a risk premium. In this framework it measures the social cost of the health inequality expressed in units of health outcomes[7].

Maximizing R in (3.1) under the budget constraint (1.1) yields at an interior optimum:

$$\frac{W(\alpha)}{\alpha}h_1'(\widehat{c}_1) = \frac{1 - W(\alpha)}{1 - \alpha}h_2'(\widehat{c}_2) \tag{3.3}$$

This optimality condition has some specific features.

a) $\widehat{c}_1 > c_1^*$ whenever $W(\alpha) > \alpha$. Whether it is expressed by a concave utility function (see previous section) or by a concave transformation of α, aversion to health inequality leads to more spending for the worst condition. This result is easily proved by noting that:

$$\frac{1 - W(\alpha)}{1 - \alpha} < 1 < \frac{W(\alpha)}{\alpha}$$

Besides when aversion to health inequality marginally increases (the already concave $W(\alpha)$ becomes more concave) so that $T(\alpha) = k(W(\alpha))$ with $k' > 0$ and $k'' < 0$, $\widehat{c}_1$ is expended further. Indeed because $T(1) = 1$ and $T(0) = 0$ as for the W function, $T(\alpha)$ is necessarily higher than $W(\alpha)$ for any α with $0 < \alpha < 1$. We then have:

$$\frac{1 - T(\alpha)}{1 - \alpha} < \frac{1 - W(\alpha)}{1 - \alpha}$$

$$\text{and } \frac{T(\alpha)}{\alpha} > \frac{W(\alpha)}{\alpha}$$

[7] Formally this social cost is equal to $(W(\alpha) - \alpha)(h_2(c_2) - h_1(c_1))$. Of course this social cost is zero when $W(\alpha) = \alpha$ i.e. under neutrality to inequality. It could also be made equal to zero if the social planner were able to choose a pair (c_1, c_2) such that $h_1(c_1) = h_2(c_2)$. However this is impossible here because of the medical technology.

so that $\widehat{\widehat{c}}_1$ is increased at the optimum and $\widehat{\widehat{c}}_2$ is reduced (by the budget constraint).

b) In the dual approach, $\widehat{\widehat{c}}_1$ is independent of an additive worsening of condition 1. Indeed the transformation of $h_1(c_1)$ into $h_1(c_1) - r$ with $r > 0$ has no impact on the optimality condition (3.3). This is obviously a weakness of the dual approach which is not observed in the "utilitarian" model. Notice besides that - as shown in appendix 3 - the dual approach makes $\widehat{\widehat{c}}_1$ sensitive to a multiplicative worsening but it does so in the wrong direction! Hence in terms of its sensitivity to a worsening of the worst condition, the dual approach is much less appealing than the utility one.

c) However, in terms of its sensitivity to the number of patients benefiting from a treatment, the dual approach does a much better job than its competitor. Indeed, to determine the impact of a change in α, rewrite the optimality condition as:

$$\frac{h_1'(\widehat{\widehat{c}}_1)}{h_2'(\widehat{\widehat{c}}_2)} = \frac{(1 - W(\alpha))\alpha}{(1 - \alpha)W(\alpha)} \tag{3.4}$$

If the right hand side of (3.4) increases (falls) with α, the concavity of the h_i functions coupled with the budget constraint implies that c_1 must fall (increase) when α increases.

Denoting the right hand side of (3.4) by M we obtain after easy transformations:

$$\frac{dM}{d\alpha} = \frac{(1 - W(\alpha))W(\alpha) - \alpha(1 - \alpha)W'(\alpha)}{(1 - \alpha)^2 W(\alpha)^2} \tag{3.5}$$

and its components are described in appendix 4.

While the analysis in appendix 4 suggests the $\frac{dM}{d\alpha}$ can be of any sign (and that it can even change sign when α moves from 0 to 1) we now produce numerical examples that illustrate this situation.

First let us consider a "power" weighting function.

$$W(\alpha) = \alpha^\theta \text{ with } 0 < \theta < 1$$

Such a function satisfies all the desirable properties of a weighting function as given in equation (3.2). With such a function:

$$\frac{dM}{d\alpha} = \frac{(1 - \alpha^\theta)\alpha^\theta - \alpha(1 - \alpha)\theta\alpha^{\theta-1}}{(1 - \alpha)^2 \alpha^{2\theta}} \tag{3.6}$$

and it is easily shown (see appendix 5) that $\frac{dM}{d\alpha}$ is everywhere positive. Hence with such a weighting function an increase in α reduces $\widehat{\widehat{c}}_1$.

It follows that such a function yields a policy recommendation that is in line with the (recent) public attention for orphan diseases that hit a very small proportion of the population and for which we believe many more medical resources should be devoted.

In order to show that the policy implication obtained with the "power" weighting function, we now consider the following piecewise weighting function:

$$\begin{aligned} W(\alpha) &= (1+k)\alpha \text{ for } 0 \leq \alpha \leq 1/2 \qquad (3.7) \\ &= (1-k)\alpha \text{ for } 1/2 \leq \alpha \leq 1 \\ \text{with } 0 &< k \leq 1 \end{aligned}$$

As shown in appendix 6, one obtain with such a function the following results:

$$\begin{aligned} \frac{dM}{d\alpha} &< 0 \text{ for } 0 < \alpha < 1/2 \\ \frac{dM}{d\alpha} &> 0 \text{ for } 1/2 < \alpha < 0 \\ \text{and } M(0) &= M(1) = 0 \end{aligned}$$

With this piecewise linear function $\widehat{\widehat{c}}_1$ increases when α first increases from 0. However when α reaches $1/2$ further increase in α reduce $\widehat{\widehat{c}}_1$.

Notice finally that the piecewise linear weighting function illustrates the fact that $\frac{dM}{d\alpha}$ change sign once in the interval $[0, 1]$.

4 Conclusion and discussions

When a social planner is inequality averse, we often implictly think

(a) that she should spend relatively more on the worst condition(s), especially when they deteriorate relatively to others.

(b) that the more inequality averse she is, the more she should spend on the worst condition(s).

(c) that she should be sensitive to the number of people who benefit from a given treatment.

Quite interestingly none of the two models used to express inequality aversion - the utility one and its dual - satisfies these 3 conditions simultaneously. Besides their strenghts and weaknesses are rather complementary: when one model performs well (badly), the other one does the reverse. This strongly suggests that a model such as the "rank dependent expected utility" (RDEU) - which contains the two previous ones as special cases - should produce results that are more in line with our intuition. Before considering such a model, one should be aware that at least three specific assumptions were made here to simplify the analysis and that their relaxation might not be easy.

1. We have assumed that whatever the amount of resources devoted to the treatment of the worst condition it remains worse than the untreated best one. However in reality this might not be the case especially when the best condition left untreated is only slightly better than the untreated worst one. The preliminary developments made for the situation where the two $h_i(c_i)$ curves intersect each other show that in this environment clear-cut results are not easily obtained and this is especially true for the dual approach in which the ranking of the outcomes is crucial. Since the difficulty appears already in the dual model it should of course also be present in the RDEU one which generalizes it.

2. We have also assumed that the effect of each treatment is known with certainty. Making the effect of treatment random should not be too difficult in the utility model because this case was already partly considered by Hoel and by Bui et *al.* However in the dual approach matters would become much more difficult because the randomness of the outcomes can change the ranking of the condition. For instance an individual with condition 1 who is lucky with its treatment may end up better than another patient with condition 2 but adverse effects of its treatment[8].

3. Finally, the - probably less penalizing - assumption that there are only two conditions should be extended to the case of n conditions with one of them corresponding to individuals in perfect health. Although at first glance such an extension may seem easy in principle, it should not be forgotten that in risk theory results that hold true in the binary case do not always easily extend to a situation with more than two states of the world.

[8] The same fact was observed by Doherty and Eeckhoudt (1995) when they introduced background risk into the analysis of insurance decisions with Yaari's dual model of choices under risk.

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A Appendix 1

As an anonymous reader of the proposal indicated, some results may be driven by the specificity of the environment. In this appendix we show indeed that in another environment conclusions might be different.

As is obvious from section 1, we assume there that c_1 and c_2 are treatment cost per patient while B is the total budget available to treat all the sick patients. Hence the budget constraint is written as in (1.1). However consider now that B is the total budget available to finance research either on disease 1 or on disease 2. In this case the budget constraint becomes:

$$B_1 + B_2 \leq 0 \tag{A.1.1}$$

where the cost for each disease (B_i) is independent of the number of patients affected by the disease.

Then in the absence of equity considerations, the objective function becomes:

$$\text{Max}_{B_1} \;\; \alpha h_1(B_1) + (1-\alpha)h_2(B - B_1) \tag{A.1.1}$$

Notice that in the present environment the medical benefits of the research effort depend upon the number of patients affected. It is then easy to see that the optimal value of B_1^* and B_2^* depend upon the value of α, contrarily to the result obtained in section 1 (eq. (1.3)).

B Appendix 2

A deterioration of condition 1 could be formalized through the multiplicative coefficient θ in

$$\theta h_1(c_1) \text{ with } 0 < \theta < 1 \tag{A.2.1}$$

A fall in θ indicates a worsening of disease 1. In this case the objective function is:

$$\text{Max} \;\; R = \alpha U(\theta h_1(c_1) - r) + (1-\alpha)U(h_2(c_2))$$

so that the first order condition is:

$$\alpha U'(\theta h_1(\widehat{c}_1))\theta h_1'(\widehat{c}_1) = (1-\alpha)U'(h_2(\widehat{c}_2))h_2'(\widehat{c}_2)$$

Then one easily obtains that

$$\text{Sign } (\frac{d\widehat{c}_1}{d\theta}) = \text{ Sign}(\frac{d}{d\theta}(U'(\theta h_1(\widehat{c}_1))\theta)) \tag{A.2.2}$$

The derivative of the right hand side of (A.2.2) is

$$U'(\theta h_1(\widehat{c}_1)) + \theta h_1(\widehat{c}_1)U''(\theta h_1(\widehat{c}_1))$$

and this derivative is negative (positive) if:

$$1 + \theta h_1(\widehat{c}_1)\frac{U''}{U'}(\theta h_1(\widehat{c}_1)) \text{ is negative (positive)}$$

In this last expression one recognizes the equivalent of the coefficient of the relative risk aversion of U.

Hence when a multiplicative deterioration of condition 1 occurs, $\widehat{c}_1$ does not necessarily increase. The intuition behind the result is simple: in the additive case the deterioration has no impact on the marginal productivity of the treatment so that only the concern for health inequality matters. For a multiplicative shift there are two effects of a fall in θ: as for the additive case, there is an incentive to increase c_1 to reduce inequality but at the same time since the marginal product of c_1 falls it tends to be reduced for efficiency reasons.

If we interpret $\theta h_1(\widehat{c}_1)\frac{U''}{U'}(\theta h_1(\widehat{c}_1))$ as a coefficient of relative inequality aversion (by analogy with the notion of relative risk aversion) our result states that inequality aversion should be sufficiently high for a fall in θ to induce an increase in $\widehat{c}_1$.

C Appendix 3

With a multiplicative worsening of condition 1 the optimality condition becomes:

$$\frac{W(\alpha)\theta}{\alpha} h_1'(\widehat{\widehat{c}}_1) = \frac{1 - W(\alpha)}{1-\alpha} h_2'(\widehat{\widehat{c}}_2) \tag{A.3.1}$$

with $0 < \theta < 1$.

When θ falls $h_1'(\widehat{\widehat{c}}_1)$ must increase to restore equilibrium so that $\widehat{\widehat{c}}_1$ falls. This result is already obtained for an inequality neutral planner because the fall in θ makes the treatment 1 less efficient at the margin. Intuition suggests that this effect should be mitigated when inequality aversion prevails. Unfortunately this does not happen when this aversion is expressed through the dual theory. Quite to the contrary because with the dual theory θ is multiplied by a number $\frac{W(\alpha)}{\alpha}$ that is larger than unity so that the effect of the fall in θ is amplified by the inequality aversion! This negative result should not be surprising in light of the discussion of appendix 2: because the dual theory implies linearity of the objective function in terms of the health outcomes it has an index of relative inequality aversion equal to zero, far below the threshold of 1 discussed in appendix 2.

D Appendix 4

In order to show that the numerator in the expression on the right hand side of (3.5) can be of any sign (while the denominator is positive) we consider first the properties of the function $\alpha(1-\alpha)$ and $W(\alpha)(1-W(\alpha)$ respectively.

The function $\alpha(1-\alpha)$ is concave in α and it reaches its maximum at its maximum at $\alpha = 1/2$ where its value amounts to 1/4.

The function $Z(\alpha) = W(\alpha)(1 - W(\alpha)$ is less easily characterized. Its first and second derivatives are respectively:

$$\frac{\delta Z}{\delta \alpha} = W'(\alpha)(1 - 2W(\alpha)) \tag{A.4.1}$$

and

$$\frac{\delta^2 Z}{\delta \alpha^2} = W''(\alpha) - 2W''(\alpha)W(\alpha) - 2(W'(\alpha))^2 = W''(\alpha)(1-2W(\alpha)) - 2(W'(\alpha))^2 \tag{A.4.2}$$

From (A.4.1) we notice that $Z(\alpha)$ is maximum at $W(\alpha) = 1/2$ that is at any α value (denoted $\widehat{\alpha}$) inferior to 1/2. Indeed since $W(\alpha)$ is everywhere increasing in α, it turns out that for $\alpha < \widehat{\alpha}$, $W(\alpha)$ is smaller than 1/2 and $\frac{\delta Z}{\delta \alpha}$ is positive while it is always negative for any value α in the interval $[\widehat{\alpha}, 1]$ because in that interval $W(\alpha) < 1/2$.

Besides, because of (A.4.2) $Z(\alpha)$ is concave in α for any $\alpha < \widehat{\alpha}$. However when α exceeds $\widehat{\alpha}$, $Z(\alpha)$ can be concave, convex or linear since $\frac{\delta^2 Z}{\delta \alpha^2}$ becomes sign ambiguous.

Finally, at $\alpha = \widehat{\alpha}$, $Z(\alpha)$ reaches its maximum, the value of which is 1/4. Hence we get figure 1 where the curve represents the $Z(\alpha)$ function.

Observing then that in the numerator of (3.5) $\alpha(1-\alpha)$ is multiplied by $W'(\alpha)$ which is larger than unity for small α's and lower than unity at high values of α, it is obvious that $Z(\alpha)$ can be positive or negative and that it can even change signs many times.

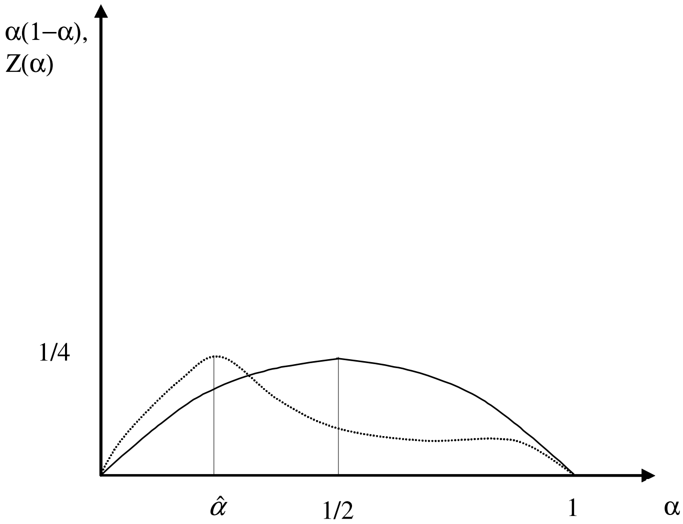

Figure 1:

E Appendix 5

To analyse the sign of $\frac{dM}{d\alpha}$ notice first that it is the same as that of the numerator in (3.6) since the denominator is always positive.

The numerator of (3.6) can be written as:

$$\alpha^{\theta}\left[1-\alpha^{\theta}-\theta+\alpha\theta\right] \tag{A.5.1}$$

The expression in brackets - that we denote $g(\alpha)$ - has the following properties:

$$g(0) = 1-\theta$$

$$g(1) = 0$$

and

$$\frac{dg}{d\alpha} = -\theta\alpha^{\theta-1}+\theta = \theta\left(1-\frac{1}{\alpha^{1-\theta}}\right) < 0$$

Hence $g(\alpha)$ must everywhere be positive and so is $\frac{dM}{d\alpha}$.

F Appendix 6

Using the piecewise linear weighting function in (1) for the case where α is below 1/2, we obtain after some obvious simplification:

$$M = \frac{1-(1+k)\alpha}{(1-\alpha)(1+k)}$$

and easy calculation yields $\frac{dM}{d\alpha} < 0$.
For $\alpha > 1/2$, we have

$$M = \frac{(1-k)(1-\alpha)\alpha}{\alpha(1-\alpha)(1-k)+k(1-\alpha)}$$

and after some tedious algebra[9] one gets $\frac{dM}{d\alpha} > 0$ on that interval.

[9] The algebra can be simplified by showing that $\frac{d(\frac{1}{M})}{d\alpha}$ is negative.